

Large-time behavior for spherically symmetric flow of viscous polytropic gas in exterior unbounded domain with large initial data

Zhilei LIANG*

E-mail: zhilei0592@gmail.com

School of Economic Mathematics,
Southwestern University of Finance and Economics,
Chengdu 611130, P.R. China

Abstract

This paper deals with the spherically symmetric flow of compressible viscous and polytropic ideal fluid in unbounded domain exterior to a ball in $\mathbb{R}^n (n \geq 2)$. We show that the global solutions are convergent as time goes to infinity. The critical step is obtaining the point-wise bound of the specific volume $v(x, t)$ and the absolute temperature $\theta(x, t)$ from up and below both for x and t . Note that the initial data can be arbitrarily large and, compared with [14], our method applies to the spatial dimension $n = 2$. The proof is based on the elementary energy methods.

1 Introduction

We study the asymptotic behavior of spherically symmetric solutions to a polytropic ideal model of a compressible viscous gas over an unbounded exterior domain $\Pi = \{\xi \in \mathbb{R}^n : |\xi| > 1\}$, where $n \geq 2$ denotes the spatial dimension. The motion of a viscous polytropic ideal gas which can be described by the equations in Eulerian coordinates (cf. [3])

$$\begin{cases} \rho_t + \operatorname{div}(\rho \mathbf{u}) = 0, \\ \rho(\mathbf{u}_t + \mathbf{u} \cdot \nabla \mathbf{u}) + R \nabla(\rho \theta) = \mu \Delta \mathbf{u} + \nabla((\mu + \lambda) \operatorname{div} \mathbf{u}), \\ c_v \rho(\theta_t + \mathbf{u} \cdot \nabla \theta) + R \rho \theta \operatorname{div} \mathbf{u} = \kappa \Delta \theta + \lambda (\operatorname{div} \mathbf{u})^2 + 2\mu D : D, \end{cases} \quad \xi \in \Pi, t > 0. \quad (1.1)$$

Here, as usual, the unknown functions ρ, θ and $\mathbf{u} = (u_1, \dots, u_n)$ symbol the density, the absolute temperature and the velocity, respectively. R, c_v, κ are given positive constants; μ and λ are the constant viscous coefficients satisfy $\mu > 0$, $2\mu + n\lambda > 0$; and $D = D(\mathbf{u})$ is the deformation tensor,

$$D_{ij} = \frac{1}{2} (\partial_j u_i + \partial_i u_j) \quad \text{and} \quad D : D = \sum_{i,j=1}^n D_{ij}^2.$$

We shall consider the equations (1.1) supplemented with the initial and boundary conditions

$$\rho(\xi, 0) = \rho_0(\xi), \mathbf{u}(\xi, 0) = \mathbf{u}_0(\xi), \theta(\xi, 0) = \theta_0(\xi), \quad \xi \in \overline{\Pi}, \quad (1.2)$$

*The author was supported by NNSFC 11301422.

and

$$\mathbf{u}(\xi, t)|_{\xi \in \partial \Pi} = u_0(\xi), \quad \frac{\partial \theta}{\partial \nu}(\xi, t)|_{\xi \in \partial \Pi} = \theta_0(\xi), \quad t \geq 0, \quad (1.3)$$

with ν being the exterior normal vector.

If the initial functions $(\rho_0(\xi), \mathbf{u}_0(\xi), \theta_0(\xi))$ are assumed to be spherically symmetric, i.e.,

$$\rho_0(\xi) = \hat{\rho}_0(r), \quad \mathbf{u}_0(\xi) = \frac{\xi}{r} \hat{u}_0(r), \quad \theta_0(\xi) = \hat{\theta}_0(r), \quad r = |\xi| \geq 1, \quad (1.4)$$

so does the corresponding solution $(\hat{\rho}, \hat{u}, \hat{\theta})(r, t)$ because (1.1) is rotationally invariant (cf. [17]), and thereby, the equations (1.1) takes the form (ignore the " ^ ")

$$\begin{aligned} \rho_t + \frac{(r^{n-1} \rho u)_r}{r^{n-1}} &= 0, \\ \rho(u_t + u \partial_r u) + R \partial_r(\rho \theta) &= \beta \left(\frac{(r^{n-1} u)_r}{r^{n-1}} \right)_r, \\ c_v \rho(\theta_t + u \partial_r \theta) + R \rho \theta \frac{(r^{n-1} u)_r}{r^{n-1}} \\ &= \kappa \frac{(r^{n-1} \theta_r)_r}{r^{n-1}} + \lambda \left(\frac{(r^{n-1} u)_r}{r^{n-1}} \right)^2 + 2\mu (\partial_r u)^2 + 2\mu \frac{n-1}{r^2} u^2, \quad r \in (1, \infty), t > 0, \end{aligned} \quad (1.5)$$

where $\beta = 2\mu + \lambda > 0$, the initial and boundary conditions (1.2)-(1.3) become

$$\rho(r, 0) = \rho_0(r), \quad u(r, 0) = u_0(r), \quad \theta(r, 0) = \theta_0(r), \quad r \geq 1, \quad (1.6)$$

and

$$u(1, t) = 0, \quad \partial_r \theta(1, t) = 0, \quad t \geq 0. \quad (1.7)$$

For our analysis convenience, it is desirable to convert the (1.5) from the Euler coordinates (r, t) into that in Lagrangian coordinates (x, t) . Define

$$r(x, t) = r_0(x) + \int_0^t u(r(x, \tau), \tau) d\tau, \quad (1.8)$$

with

$$\int_1^{r_0(x)} y^{n-1} \rho_0(y) dy = x. \quad (1.9)$$

Using (1.8), (1.9), (1.5)₁, and the boundary condition $u(1, t) = 0$, we check for $t \geq 0$

$$\int_1^{r(x, t)} y^{n-1} \rho(y, t) dy = \int_1^{r_0(x)} y^{n-1} \rho_0(y) dy = x. \quad (1.10)$$

By this, $r = 1$ iff $x = 0$ and $r \rightarrow \infty$ iff $x \rightarrow \infty$, as long as $\rho > 0$ for all $(y, t) \in [0, \infty) \times [0, \infty)$. Moreover, it is easy to see from (1.8) and (1.10) that

$$\partial_t r(x, t) = u(r(x, t), t) \text{ and } r^{n-1}(x, t) \rho(r(x, t), t) \partial_x r(x, t) = 1. \quad (1.11)$$

Introduce new functions

$$\tilde{v}(x, t) = 1/\rho(r(x, t), t), \quad \tilde{u}(x, t) =: u(r(x, t), t), \quad \tilde{\theta}(x, t) =: \theta(r(x, t), t), \quad (1.12)$$

we express (1.5) in terms of $(\tilde{v}, \tilde{u}, \tilde{\theta})$ (denoted still by (v, u, θ) below) in variables (x, t)

$$\begin{aligned} v_t &= (r^{n-1} u)_x, \\ u_t &= r^{n-1} \sigma_x, \\ c_v \theta_t &= \kappa \left(\frac{r^{2(n-1)} \theta_x}{v} \right)_x + (r^{n-1} u)_x \sigma - 2\mu(n-1)(r^{n-2} u^2)_x, \quad x \in \Omega, t > 0, \end{aligned} \quad (1.13)$$

where $\sigma = \beta(r^{n-1}u)_x/v - R\theta/v$, $\Omega = (0, +\infty)$, the initial functions

$$v(x, 0) = v_0(x), u(x, 0) = u_0(x), \theta(x, 0) = \theta_0(x), \quad x \in \Omega, \quad (1.14)$$

the boundary and the far field behavior

$$u(0, t) = 0, \quad \partial_x \theta(0, t) = 0, \quad \lim_{x \rightarrow \infty} (v(x, t), u(x, t), \theta(x, t)) = (1, 0, 1) \quad t \geq 0. \quad (1.15)$$

In view of (1.12), the (1.8) and (1.11) is reduced to

$$r(x, t) = r_0(x) + \int_0^t u(x, s) ds, \quad r_t = u, \quad r^{n-1} r_x = v. \quad (1.16)$$

Integration of last term in (1.16) yields

$$r^n(x, t) = 1 + n \int_0^x v(y, t) dy. \quad (1.17)$$

Furthermore, it follows from [6, eq.(3.19)] that

$$r(x, t) \geq r(0, t) = 1, \quad (x, t) \in \overline{\Omega} \times [0, \infty). \quad (1.18)$$

We first state the global existence in time of (generalized) solution to the initial-boundary-value problem (1.13)-(1.15).

Theorem 1.1 (See [6]) *Assume that the initial function (v_0, u_0, θ_0) in (1.14) are compatible with the boundary conditions (1.15), and satisfy*

$$v_0 - 1, u_0, \theta_0 - 1, r^{n-1} \partial_x v_0, r^{n-1} \partial_x u_0, r^{n-1} \partial_x \theta_0 \in L^2(\Omega), \quad (1.19)$$

$$\inf_{x \in \overline{\Omega}} v_0(x) > 0 \quad \text{and} \quad \inf_{x \in \overline{\Omega}} \theta_0(x) > 0. \quad (1.20)$$

Then for any fixed $T > 0$, the problem (1.13)-(1.15) admits a unique global (large) generalized solution (v, u, θ) over $[0, T]$, with $v(x, t)$ and $\theta(x, t)$ having positive bounds from above and below (depending on T). Moreover,

$$\begin{aligned} v - 1, u, \theta - 1 &\in L^\infty(0, T; H^1(\Omega)), \quad v_t, r^{n-1} u_x, r^{n-1} \theta_x \in L^\infty(0, T; L^2(\Omega)), \\ v_{xt}, u_t, \theta_t, r^{2(n-1)} u_{xx}, r^{2(n-1)} \theta_{xx} &\in L^2(0, T; L^2(\Omega)). \end{aligned} \quad (1.21)$$

After the existence of global solution, the study of asymptotic behavior for solutions comes out naturally. The first paper in this aspect is due to Matsumura-Nishida [13], where they prove that, with initial data having a small oscillatory around a non-vacuum equilibrium, the solution (with external force and may not be symmetric) converges to the corresponding stationary solution as time tends to infinity in exterior domain of \mathbb{R}^3 . From then on, much progress have been made with smallness assumptions. See [4, 5, 9, 11, 15, 16], and the references cited therein.

In case of large initial data in bounded annulus domain, Itaya [17] showed global existence of the spherically symmetric solution to (1.1). Matsumura [12] considered the isothermal flow and proved that the stationary solution is time asymptotically stable (with large external force); also, an exponential convergence rate was obtained there.

When the exterior domains becomes unbounded, Jiang [6] obtained the global in time solutions to (1.13)-(1.15). When it comes to the large-time behavior, some difficulties arise: for example, the useful representation (see [2]) for v , the specific volume, and the imbedding inequality $L^2 \hookrightarrow L^1$ do not valid any more for unbounded domains. However, when the space dimension $n \geq 3$, Jiang [6] proved a partial result on the asymptotic behavior, precisely, he shows that $\|u\|_{L^{2j}(\Omega)}$ is zero stable as time tends into infinity, with $j \in [2, \infty)$ being an arbitrary integer. Nakamura-Nishibata [14] proved that the solutions (with external force) correspond to the stationary solution time asymptotically. But the proof in [14] still requires $n \geq 3$, which is

essential to conquer (with (3.16)) the difficulty caused by the unboundedness. Finally, we mention the progress for the one-dimensional (1D) case. By means of cut-off function, Jiang [7, 8] obtained the uniform (in x and t) bounds for $v(x, t)$, but leaves that for absolute temperature $\theta(x, t)$ open. Recently, Li and the author [10] gives a complete description on the large-time behavior of solutions to 1D Cauchy problem (1.13).

This paper concerns the spherically symmetric solutions in unbounded domain exterior to \mathbb{R}^n with spatial dimension $n \geq 2$. Our goal is to show the large-time behavior of solutions to the initial-boundary-value problem (1.13)-(1.15). The proof depends heavily on the bounds of the specific volume $v(x, t)$ and the absolute temperature $\theta(x, t)$. We adopt the idea in [2, 7] and use a local representation to derive the bound for $v(x, t)$. To get the bound for the temperature, we multiply the equations (1.13) by $(\theta - 2)_+$, because the spatial domain always keeps bounded when $\theta(x, t)$ leaves far away from the equilibrium state at any fixed time.

Our main result in current paper lies in the following theorem.

Theorem 1.2 (Large time behavior) *Assume that the initial data defined in (1.14) are compatible with boundary conditions (1.15) and satisfy (1.19)-(1.20). Let (v, u, θ) be the (unique) generalized solution to (1.13)-(1.15) described in Theorem 1.1 which satisfies (1.21). Then there exists a positive constant C depending only on $\mu, \lambda, R, c_v, \kappa, n$, and the initial data, such that*

$$\begin{aligned} & \sup_{t \in [0, \infty)} (\|(v - 1, u, \theta - 1)(\cdot, t)\|_{L^2(\Omega)} + \|r^{n-1}(v_x, u_x, \theta_x)(\cdot, t)\|_{L^2(\Omega)}) \\ & + \int_0^\infty \left(\|r^{2(n-1)}u_{xx}\|_{L^2(\Omega)}^2 + \|r^{2(n-1)}\theta_{xx}\|_{L^2(\Omega)}^2 \right) dt \leq C, \end{aligned} \quad (1.22)$$

and

$$C^{-1} \leq v(x, t), \quad \theta(x, t) \leq C, \quad \forall (x, t) \in \overline{\Omega} \times [0, \infty). \quad (1.23)$$

Moreover, the following asymptotic behavior holds

$$\lim_{t \rightarrow \infty} \|(v - 1, u, \theta - 1)(\cdot, t)\|_{C(\overline{\Omega})} = 0. \quad (1.24)$$

A few remarks are in order:

Remark 1.1 *In comparison with [6, 14], the (1.23) in Theorem 1.2 is valid for $n = 2$. Moreover, we do not need any type of smallness assumptions on the initial data.*

Remark 1.2 *The same conclusion as Theorem 1.2 holds true if the boundary condition (1.15) is replaced by*

$$u(0, t) = 0, \quad \theta(0, t) = 1, \quad t \geq 0.$$

Throughout this paper, $C(\overline{\Omega})$, $L^p(\Omega)$ and $H^1(\Omega)$ denote the usual Sobolev spaces. See, for example, the definitions in [1]. The same letter C symbols a positive generic constant which may rely on $\mu, \lambda, R, c_v, \kappa, n$, and the initial data, but does not depend on the time t . Particularly, we use $C(\alpha)$ to emphasize that C depends on α .

The remainder sections are arranged as follows:

In section 2, some known Lemmas and facts are collected for our usage.

In sections 3, we use a local representation to get the uniform bound for $v(x, t)$ from above and below, and in section 4, the uniform upper bound for $\theta(x, t)$ is derived by means of elaborate energy computation.

We give the L^2 -norm estimates for derivatives of the solutions in section 5, and complete the proof of Theorem 1.2 in the final section 6.

2 Preliminaries

The first lemma provides the basic energy estimate.

Lemma 2.1 *The solution (v, u, θ) obtained in Theorem 1.1 satisfies for all $t \geq 0$*

$$\int_{\Omega} U(x, t) + \int_0^t \int_{\Omega} \left(\frac{vu^2}{r^2\theta} + \frac{r^{2(n-1)}u_x^2}{v\theta} + \frac{(r^{n-1}u)_x^2}{v\theta} + \frac{r^{2(n-1)}\theta_x^2}{v\theta^2} \right) \leq C, \quad (2.1)$$

where

$$U(x, t) = \left(R(v - \ln v - 1) + \frac{1}{2}u^2 + c_v(\theta - \ln \theta - 1) \right) (x, t).$$

Proof. Multiplying (1.13)₁ by $R(1-v^{-1})$, (1.13)₂ by u , (1.13)₃ by $(1-\theta^{-1})$, respectively, adding them together, we arrive at

$$\begin{aligned} U_t + \left(\beta \frac{(r^{n-1}u)_x^2}{v\theta} - 2\mu(n-1) \frac{(r^{n-2}u^2)_x}{\theta} + \kappa \frac{r^{2(n-1)}\theta_x^2}{v\theta^2} \right) \\ = \left\{ \beta \frac{r^{n-1}u(r^{n-1}u)_x}{v} + Rr^{n-1}u \left(1 - \frac{\theta}{v} \right) + \kappa \frac{r^{2(n-1)}\theta_x(\theta-1)}{v\theta} - 2\mu(n-1)(r^{n-2}u^2) \right\}_x. \end{aligned} \quad (2.2)$$

Utilizing (1.16), a careful calculation (see [14, Lemma 3.1] for detail) shows there exists a positive constant C such that

$$\beta(r^{n-1}u)_x^2 - 2\mu(n-1)v(r^{n-2}u^2)_x \geq C \left(r^{-2}v^2u^2 + r^{2(n-1)}u_x^2 \right). \quad (2.3)$$

With (2.3), integrating (2.2), using integration by parts, Taylor theorem, initial conditions (1.19)-(1.20), we obtain

$$\int_{\Omega} U(x, t) + \int_0^t \int_{\Omega} \left(\frac{vu^2}{r^2\theta} + \frac{r^{2(n-1)}u_x^2}{v\theta} + \frac{r^{2(n-1)}\theta_x^2}{v\theta^2} \right) \leq C. \quad (2.4)$$

Again using (1.16) we compute

$$(r^{n-1}u)_x = r^{n-1}u_x + (n-1)r^{-1}vu, \quad (2.5)$$

which together with (2.4) yields

$$\int_0^t \int_{\Omega} \frac{(r^{n-1}u)_x^2}{v\theta} \leq C.$$

This inequality plus (2.4) yields (2.1). \square

Having (2.1) in hand, we use Jensen inequality and check that

$$\int_k^{k+1} v - \ln \int_k^{k+1} v - 1, \quad \int_k^{k+1} \theta - \ln \int_k^{k+1} \theta - 1 \leq C, \quad k = 0, 1, 2, \dots$$

This implies from mean value theorem

$$0 < \alpha_1 \leq v(a_k(t), t) = \int_k^{k+1} v(x, t), \quad \theta(b_k(t), t) = \int_k^{k+1} \theta(x, t) \leq \alpha_2 < \infty, \quad (2.6)$$

where α_1, α_2 are two positive roots of the equation $y - \ln y - 1 = C$.

3 Uniform bounds of $v(x, t)$

Lemma 3.1 *Let (v, u, θ) be a solution described in Theorem 1.1. Then it satisfies*

$$C^{-1} \leq v(x, t) \leq C, \quad (x, t) \in \bar{\Omega} \times [0, +\infty). \quad (3.1)$$

Proof. The strategy is adopt some ideas in [2, 7, 8] to localize the problem.

Local representation for $v(x, t)$:

Define the cut-off function

$$\varphi(x) = \begin{cases} 1, & y \leq k; \\ k+1-x, & k \leq y \leq k+1; \\ 0, & y \geq k+1. \end{cases}$$

Utilizing (1.16), we multiply (1.13)₂ by φ and compute

$$(\varphi r^{1-n} u)_t + (n-1)\varphi r^{-n} u^2 = (\sigma \varphi)_x - \sigma \varphi'. \quad (3.2)$$

Let $x \in I = (k-2, k) \cap \Omega$ with $k \in \mathbb{N}_+$, integrating (3.2) over $(x, +\infty)$ and using (1.13)₁ lead to

$$\begin{aligned} -\partial_t \int_x^\infty \varphi r^{1-n} u - (n-1) \int_x^\infty \varphi r^{-n} u^2 &= \sigma + \int_x^\infty \varphi' \sigma \\ &= \beta (\ln v)_t - R \frac{\theta}{v} - \int_k^{k+1} \sigma. \end{aligned}$$

Integration of it in time shows

$$\begin{aligned} \int_x^\infty \varphi (r_0^{1-n} u_0 - r^{1-n} u) - (n-1) \int_0^t \int_x^\infty \varphi r^{-n} u^2 \\ = \beta \ln \frac{v(x, t)}{v_0} - R \int_0^t \frac{\theta}{v} - \int_0^t \int_k^{k+1} \sigma, \end{aligned}$$

which gives after taken the exponential

$$\frac{1}{v(x, t)} \exp \left\{ \frac{R}{\beta} \int_0^t \frac{\theta}{v} \right\} = \frac{1}{B(x, t) Y(t)}, \quad x \in I, t \geq 0, \quad (3.3)$$

where

$$B(x, t) = v_0 \exp \left\{ \frac{1}{\beta} \int_x^\infty \varphi (r_0^{1-n} u_0 - r^{1-n} u) \right\}$$

and

$$Y(t) = \exp \left\{ \frac{1}{\beta} \left(\int_0^t \int_k^{k+1} \sigma - (n-1) \int_0^t \int_x^\infty \varphi r^{-n} u^2 \right) \right\}.$$

Integrating (3.3) after multiplied by $R\theta/\beta$ arrives at

$$\exp \left\{ \frac{R}{\beta} \int_0^t \frac{\theta}{v} d\tau \right\} = 1 + \frac{R}{\beta} \int_0^t \frac{\theta(x, \tau) B(x, \tau) Y(\tau)}{B(x, \tau) Y(\tau)} d\tau, \quad x \in I, \quad (3.4)$$

which again with (3.3) yields

$$v(x, t) = B(x, t) Y(t) + \frac{R}{\beta} \int_0^t \frac{\theta(x, \tau) B(x, \tau) Y(\tau)}{B(x, \tau) Y(\tau)} d\tau, \quad x \in I, t \geq 0. \quad (3.5)$$

Estimate for $B(x, t)$ and $Y(t)$:

Clearly, it follows from (1.18) and (2.1) that for $x \in I$

$$\left| \int_x^\infty \varphi(r_0^{1-n} u_0 - r^{1-n} u) dy \right| \leq C \|u_0\|_{L^2(k-2, k+1)} + C \|u\|_{L^2(k-2, k+1)} \leq C,$$

and therefrom,

$$C^{-1} \leq B(x, t) \leq C. \quad (3.6)$$

Next, we following in [7] to estimate $Y(t)$. Making use of (2.6) and (1.18), one has for $x \in [k, k+1]$ and $0 \leq s < t$

$$\begin{aligned} \left| \int_s^t \int_{b_k}^x \frac{\theta_x}{\theta} \right| &\leq \int_s^t \left(\int_k^{k+1} \frac{\theta_x^2}{v\theta^2} \right)^{1/2} \left(\int_k^{k+1} v(y, \tau) \right)^{1/2} \\ &\leq C \alpha_2^{1/2} \int_s^t \left(\int_k^{k+1} \frac{r^{2(n-1)} \theta_x^2}{v\theta^2} \right)^{1/2} \leq C(\alpha_2) + (t-s) \ln 2. \end{aligned}$$

By this we use Jensen inequality to estimate

$$\begin{aligned} \int_s^t \theta(x, \tau) &= \int_s^t \exp\{\ln \theta(x, \tau)\} \\ &\geq (t-s) \exp \left\{ \frac{1}{(t-s)} \int_s^t \ln \theta(x, \tau) \right\} \\ &= (t-s) \exp \left\{ \frac{1}{(t-s)} \left[\int_s^t \int_{b_k}^x \frac{\theta_x}{\theta} + \int_s^t \ln \theta(b_k, \tau) \right] \right\} \\ &\geq (t-s) \exp \left\{ \frac{-1}{(t-s)} \left| \int_s^t \int_{b_k}^x \frac{\theta_x}{\theta} \right| + \ln \alpha_1 \right\} \\ &\geq \frac{\alpha_1}{2} (t-s) \exp \left\{ \frac{-C(\alpha_2)}{(t-s)} \right\}, \end{aligned}$$

whence,

$$-\int_s^t \inf_{x \in [k, k+1]} \theta(\cdot, \tau) \leq \begin{cases} 0, & 0 \leq t-s \leq 1, \\ -C(t-s), & 1 \leq t-s. \end{cases} \quad (3.7)$$

By virtue of (2.1), (2.6), (3.7), Jensen inequality, we have

$$\begin{aligned} \int_s^t \int_k^{k+1} \sigma - (n-1) \int_0^t \int_x^\infty \varphi r^{-n} u^2 \\ \leq \int_s^t \int_k^{k+1} \sigma = \int_s^t \int_k^{k+1} \left(\beta \frac{(r^{n-1} u)_x}{v} - R \frac{\theta}{v} \right) \\ \leq C \int_s^t \int_k^{k+1} \frac{(r^{n-1} u)_x^2}{v\theta} - \frac{R}{2} \int_0^t \int_k^{k+1} \frac{\theta}{v} \\ \leq C - \frac{R}{2} \int_s^t \inf \theta(\cdot, \tau) \left(\int_k^{k+1} v \right)^{-1} \\ \leq C - C \int_s^t \inf \theta(\cdot, \tau) \leq C - C(t-s). \end{aligned}$$

Therefore,

$$0 \leq Y(t)/Y(s) \leq C \exp\{-C(t-s)\}, \quad 0 \leq s < t. \quad (3.8)$$

Uniform bounds of $v(x, t)$ from up and below:

In terms of (3.6) and (3.8), we deduce from (3.5) that for $x \in I$

$$v(x, t) \leq C + C \int_0^t \theta(x, s) \exp\{-C(t-s)\} ds. \quad (3.9)$$

Observe from (1.18) and (2.6) that for $x \in [k, k+1]$

$$\begin{aligned} \left| \sqrt{\theta(x, t)} - \sqrt{\theta(b_k(t), t)} \right| &\leq \int_k^{k+1} \left| \frac{\theta_x}{\sqrt{\theta}} \right| \leq \left(\int_k^{k+1} \frac{\theta_x^2}{v\theta^2} \right)^{1/2} \left(\int_k^{k+1} v\theta \right)^{1/2} \\ &\leq C\sqrt{\alpha_2} \max_{x \in [k, k+1]} \sqrt{v} \left(\int_k^{k+1} \frac{r^{2(n-1)}\theta_x^2}{v\theta^2} \right)^{1/2}, \end{aligned}$$

which implies that

$$\frac{\alpha_1}{2} - \alpha_2 f(t) \max_{x \in [k, k+1]} v(\cdot, t) \leq \theta(x, t) \leq 2\alpha_2 + 2\alpha_2 f(t) \max_{x \in [k, k+1]} v(\cdot, t), \quad (3.10)$$

where

$$f(t) = \int_{\Omega} \frac{r^{2(n-1)}\theta_x^2}{v\theta^2}. \quad (3.11)$$

Inserting (3.10) into (3.9) yields

$$v(x, t) \leq C + C \int_0^t f(\tau) \max_{x \in [k, k+1]} v(\cdot, \tau). \quad (3.12)$$

Recall (2.1), exploiting Gronwall inequality to (3.12) concludes

$$v(x, t) \leq C, \quad (x, t) \in [k, k+1] \times [0, \infty). \quad (3.13)$$

Integrate (3.5) over $[k, k+1]$, use (2.6), we infer

$$\alpha_1 \leq C \exp\{-Ct\} + C \int_0^t \frac{Y(t)}{Y(\tau)} d\tau.$$

which, along with (3.6), (3.8), (3.10), (3.13), deduces from (3.5) that for $x \in [k, k+1]$

$$\begin{aligned} v(x, t) &\geq C \int_0^t \theta(x, \tau) \frac{Y(t)}{Y(\tau)} d\tau \\ &\geq C \int_0^t \frac{Y(t)}{Y(\tau)} d\tau - C \int_0^t f(s) \frac{Y(t)}{Y(\tau)} d\tau \\ &\geq C - C \exp\{-Ct\} - C \left(\int_0^{t/2} + \int_{t/2}^t \right) f(\tau) \exp\{-C(t-\tau)\} d\tau \\ &\geq C - C \exp\{-Ct/2\} - \int_{t/2}^t f(\tau) d\tau \\ &\geq C, \end{aligned} \quad (3.14)$$

as long as $t \geq T_0$ for some large T_0 . On the other hand, it satisfies from [6, eq.(4.9)] that

$$v(x, t) \geq C(T_0), \quad (x, t) \in \overline{\Omega} \times [0, T_0]. \quad (3.15)$$

Notice that the C is independent of k , the proof ends up with (3.13), (3.14) and (5.2) if the integers k traverses \mathbb{N}_+ . \square

Corollary 3.2 *Inequalities (1.17) and (3.1) ensure that*

$$C^{-1}(1+x) \leq r^n(x, t) \leq C(1+x), \quad (3.16)$$

where the positive constant C independent of either x or t .

Remark 3.1 *With the aid of (3.16), the validity of (3.1) has been proven by Jiang in [6] for $n = 3$.*

4 Uniform bound for $\theta(x, t)$ from above

The following lemma plays a critical role in deriving the upper bound for θ .

Lemma 4.1 *Let (v, u, θ) be the solution described in Theorem 1.1. Then it holds that*

$$\int_{\Omega} [(\theta - 1)^2 + u^4](x, t) + \int_0^t \int_{\Omega} [(1 + \theta + u^2)(r^{n-1}u)_x^2 + r^{2(n-1)}\theta_x^2] \leq C, \quad (4.1)$$

where the C is independent of t .

Proof. First notice that the set

$$\Omega_a(t) = \{x \in \Omega : \theta(x, t) > a > 1\}$$

is uniformly bounded in time, that is, for any $t \in [0, \infty)$

$$\text{meas } \Omega_a(t) \leq \int_{\Omega_a(t)} \leq C(a) \int_{\Omega_a(t)} c_v(\theta - \ln \theta - 1) \leq C(a), \quad (4.2)$$

by (2.1). This, together with (2.6), yields

$$\int_{\Omega_a(t)} \theta(x, t) \leq C(a). \quad (4.3)$$

The proof is broken into several steps.

Step 1. Multiplied by $(\theta - 2)_+$ with $(\theta - 2)_+ = \max\{0, \theta - 2\}$, it yields from (1.13)₃ that

$$\begin{aligned} & \frac{c_v}{2} \int_{\Omega} (\theta - 2)_+^2(x, t) + \kappa \int_0^t \int_{\Omega} \frac{r^{2(n-1)} |\partial_x(\theta - 2)_+|^2}{v} \\ &= \frac{c_v}{2} \int_{\Omega} (\theta_0 - 2)_+^2 + 2\mu(n-1) \int_0^t \int_{\Omega} r^{n-2} u^2 \partial_x(\theta - 2)_+ \\ & \quad + \beta \int_0^t \int_{\Omega} \frac{(r^{n-1}u)_x^2}{v} (\theta - 2)_+ - \int_0^t \int_{\Omega} R \frac{\theta}{v} (r^{n-1}u)_x (\theta - 2)_+. \end{aligned} \quad (4.4)$$

If multiply (1.13)₂ by $2u(\theta - 2)_+$, we discover

$$\begin{aligned} & 2 \int_0^t \int_{\Omega} u_t u (\theta - 2)_+ + 2\beta \int_0^t \int_{\Omega} \frac{(r^{n-1}u)_x^2}{v} (\theta - 2)_+ \\ &= -2\beta \int_0^t \int_{\Omega} \frac{(r^{n-1}u)_x}{v} r^{n-1} u \partial_x(\theta - 2)_+ - 2 \int_0^t \int_{\Omega} R \left(\frac{\theta}{v} \right)_x r^{n-1} u (\theta - 2)_+. \end{aligned} \quad (4.5)$$

Putting (4.4) and (4.5) together receives

$$\begin{aligned} & \frac{c_v}{2} \int_{\Omega} (\theta - 2)_+^2 + \beta \int_0^t \int_{\Omega} \frac{(r^{n-1}u)_x^2}{v} (\theta - 2)_+ + \kappa \int_0^t \int_{\Omega} \frac{r^{2(n-1)} |\partial_x(\theta - 2)_+|^2}{v} \\ &= \frac{c_v}{2} \int_{\Omega} (\theta_0 - 2)_+^2 + 2\mu(n-1) \int_0^t \int_{\Omega} r^{n-2} u^2 \partial_x(\theta - 2)_+ \\ & \quad - 2\beta \int_0^t \int_{\Omega} \frac{(r^{n-1}u)_x}{v} r^{n-1} u \partial_x(\theta - 2)_+ + 2 \int_0^t \int_{\Omega} R \frac{\theta}{v} r^{n-1} u \partial_x(\theta - 2)_+ \\ & \quad + \int_0^t \int_{\Omega} R \frac{\theta}{v} (r^{n-1}u)_x (\theta - 2)_+ - 2 \int_0^t \int_{\Omega} u_t u (\theta - 2)_+ \\ &= \frac{c_v}{2} \int_{\Omega} (\theta_0 - 2)_+^2 + \sum_{i=1}^5 I_i. \end{aligned} \quad (4.6)$$

We estimate I_i ($i = 1 \sim 5$) as follows: By Cauchy-Schwarz inequality and (1.18), it has

$$\begin{aligned} I_1 &= 2\mu(n-1) \int_0^t \int_{\Omega} r^{n-2} u^2 \partial_x(\theta-2)_+ \\ &\leq \frac{\kappa}{8} \int_0^t \int_{\Omega} \frac{r^{2(n-1)} |\partial_x(\theta-2)_+|^2}{v} + C \int_0^t \int_{\Omega_2(t)} u^4 \end{aligned}$$

and

$$\begin{aligned} I_2 &= -2\beta \int_0^t \int_{\Omega} \frac{(r^{n-1}u)_x r^{n-1} u \partial_x(\theta-2)_+}{v} \\ &\leq \frac{\kappa}{8} \int_0^t \int_{\Omega} \frac{r^{2(n-1)} |\partial_x(\theta-2)_+|^2}{v} + C \int_0^t \int_{\Omega} (r^{n-1}u)_x^2 u^2. \end{aligned}$$

The third term

$$\begin{aligned} I_3 &= 2 \int_0^t \int_{\Omega} R \frac{\theta}{v} r^{n-1} u \partial_x(\theta-2)_+ \\ &= 2 \int_0^t \int_{\Omega} R \frac{(\theta-2)_+}{v} r^{n-1} u \partial_x(\theta-2)_+ + 4 \int_0^t \int_{\Omega} R \frac{r^{n-1} u}{v} \partial_x(\theta-2)_+ \\ &= 2 \int_0^t \int_{\Omega} R \frac{(\theta-2)_+}{v} r^{n-1} u \partial_x(\theta-2)_+ \\ &\quad - 4 \int_0^t \int_{\Omega} R \left(\frac{1}{v} \right)_x r^{n-1} u (\theta-2)_+ - 4 \int_0^t \int_{\Omega} R \frac{(r^{n-1}u)_x}{v} (\theta-2)_+ \\ &= 2 \int_0^t \int_{\Omega} R \frac{(\theta-2)_+}{v} r^{n-1} u \partial_x(\theta-2)_+ + I_3^1 + I_3^2. \end{aligned} \tag{4.7}$$

For one hand,

$$\begin{aligned} I_3^1 &= -4 \int_0^t \int_{\Omega} R \left(\frac{1}{v} \right)_x r^{n-1} u (\theta-2)_+ \\ &= 4 \int_0^t \int_{\Omega} R \frac{1-v}{v} (r^{n-1}u)_x (\theta-2)_+ + 4 \int_0^t \int_{\Omega} R \frac{1-v}{v} r^{n-1} u \partial_x(\theta-2)_+ \\ &\leq \beta \int_0^t \int_{\Omega} \frac{(r^{n-1}u)_x^2}{v} + C(\beta) \int_0^t \int_{\Omega} (v-1)^2 (\theta-2)_+^2 \\ &\quad + \frac{\kappa}{16} \int_0^t \int_{\Omega} \frac{r^{2(n-1)} |\partial_x(\theta-2)_+|^2}{v} + C(\kappa) \int_0^t \int_{\Omega_2(t)} (v-1)^2 u^2 (\theta-1), \end{aligned} \tag{4.8}$$

where the compensated term $(\theta(x, t) - 1) \geq 1$ in $\Omega_2(t)$.

Thanks to (2.1), (3.1), (1.18), and the fact $\theta^{-1}(x, t) \leq 1/2$ in $\Omega_2(t)$, it satisfies

$$\left| \int_0^t \int_{\Omega_2(t)} \frac{r^{n-2} u^2 \theta_x}{\theta^2} \right| \leq C \int_0^t \int_{\Omega_2(t)} \left(\frac{r^{2(n-1)} \theta_x^2}{v \theta^2} + \frac{u^4}{r^2 \theta^2} \right) \leq C + C \int_0^t \max_{x \in \Omega_2(t)} u^4. \tag{4.9}$$

From (1.13)₃ we compute

$$\begin{aligned} I_3^2 &= -4 \int_0^t \int_{\Omega} R \frac{(r^{n-1}u)_x}{v} (\theta-2)_+ \\ &= 4 \int_0^t \int_{\Omega} \left[c_v \theta_t - \kappa \left(\frac{r^{2(n-1)} \theta_x}{v} \right)_x - \beta \frac{(r^{n-1}u)_x^2}{v} + 2\mu(n-1) (r^{n-2} u^2)_x \right] \left(1 - \frac{2}{\theta} \right)_+ \\ &= 4c_v \int_0^t \int_{\Omega} \theta_t \left(1 - \frac{2}{\theta} \right)_+ + 8 \int_0^t \int_{\Omega_2(t)} \left(\kappa \frac{r^{2(n-1)} \theta_x^2}{v \theta^2} + \beta \frac{(r^{n-1}u)_x^2}{v \theta} \right) \\ &\quad - 4\beta \int_0^t \int_{\Omega_2(t)} \frac{(r^{n-1}u)_x^2}{v} - 16\mu(n-1) \int_0^t \int_{\Omega_2(t)} \frac{r^{n-2} u^2 \theta_x}{\theta^2} \\ &\leq C + C \int_0^t \max_{x \in \Omega_2(t)} u^4 - 4\beta \int_0^t \int_{\Omega} \frac{(r^{n-1}u)_x^2}{v}, \end{aligned} \tag{4.10}$$

where in the last inequality is valid because of (2.1), (4.9), and the following two inequalities

$$\begin{aligned} \int_0^t \int_{\Omega} \theta_t \left(1 - \frac{2}{\theta}\right)_+ &\leq \int_0^t \int_{\Omega} \theta_t \left(1 - \frac{2}{\theta}\right)_+ \\ &= \int_{\Omega} (\theta - 2 \ln \theta - 2(1 - \ln 2))_+ - \int_{\Omega} (\theta_0 - 2 \ln \theta_0 - 2(1 - \ln 2))_+ \\ &\leq C \end{aligned}$$

and

$$\begin{aligned} - \int_0^t \int_{\Omega_2(t)} \frac{(r^{n-1}u)_x^2}{v} &= \int_0^t \int_{\Omega \setminus \Omega_2(t)} \frac{(r^{n-1}u)_x^2}{v} - \int_0^t \int_{\Omega} \frac{(r^{n-1}u)_x^2}{v} \\ &\leq 2 \int_0^t \int_{\Omega \setminus \Omega_2(t)} \frac{(r^{n-1}u)_x^2}{v\theta} - \int_0^t \int_{\Omega} \frac{(r^{n-1}u)_x^2}{v} \\ &\leq C - \int_0^t \int_{\Omega} \frac{(r^{n-1}u)_x^2}{v}. \end{aligned}$$

Substituting (4.8) and (4.10) into (4.7) and utilizing Cauchy-Schwarz inequality guarantee that

$$\begin{aligned} I_3 &\leq C + \frac{\kappa}{8} \int_0^t \int_{\Omega} \frac{r^{2(n-1)} |\partial_x(\theta - 2)_+|^2}{v} - 3\beta \int_0^t \int_{\Omega_2(t)} \frac{(r^{n-1}u)_x^2}{v} \\ &\quad + C \int_0^t \left[\max_{x \in \Omega} (\theta - 2)_+^2 + \max_{x \in \Omega_2(t)} (u^4 + (\theta - 1)^2) \right], \end{aligned}$$

where we have used the inequality $\int_{\Omega} (u^2 + (v - 1)^2) \leq C$ which is valid due to (2.1) and (3.1).

Young inequality and (4.3) imply that

$$\begin{aligned} I_4 &= \int_0^t \int_{\Omega} R \frac{\theta}{v} (r^{n-1}u)_x (\theta - 2)_+ = \varepsilon \int_0^t \int_{\Omega} \theta (r^{n-1}u)_x^2 + C(\varepsilon) \int_0^t \int_{\Omega_2(t)} \theta (\theta - 2)_+^2 \\ &\leq \varepsilon \int_0^t \int_{\Omega} \theta (r^{n-1}u)_x^2 + C(\varepsilon) \int_0^t \max_{x \in \Omega_2(t)} (\theta - 2)_+^2, \end{aligned}$$

with small positive constant ε will be determined later.

Integration by parts and (1.13)₃ lead to

$$\begin{aligned} I_5 &= -2 \int_0^t \int_{\Omega} u_t u (\theta - 2)_+ \\ &\leq \int_{\Omega} u_0^2 (\theta_0 - 2)_+ + \int_0^t \int_{\Omega_2(t)} u^2 \partial_t \theta \\ &\leq C + \frac{\kappa}{c_v} \int_0^t \int_{\Omega_2(t)} u^2 \left(\frac{r^{2(n-1)} \theta_x}{v} \right)_x + c_v^{-1} \int_0^t \int_{\Omega_2(t)} u^2 \tilde{R}, \end{aligned} \tag{4.11}$$

with

$$\tilde{R} = \beta \frac{(r^{n-1}u)_x^2}{v} - \frac{R(\theta - 2)_+}{v} (r^{n-1}u)_x - \frac{2R}{v} (r^{n-1}u)_x - 2\mu(n-1)(r^{n-2}u^2)_x.$$

Define

$$\text{sgn}_{\eta} s = \begin{cases} 1, & s > \eta, \\ s/\eta, & 0 \leq s \leq \eta, \\ 0, & s \leq 0, \end{cases}$$

we use Lebesgue dominated convergence theorem to estimate

$$\begin{aligned}
& \int_0^t \int_{\Omega_2(t)} u^2 \left(\frac{r^{2(n-1)} \theta_x}{v} \right)_x \\
&= \lim_{\eta \rightarrow 0+} \int_0^t \int_{\Omega} u^2 \operatorname{sgn}_{\eta}(\theta - 2) \left(\frac{r^{2(n-1)} \theta_x}{v} \right)_x \\
&= - \lim_{\eta \rightarrow 0+} \int_0^t \int_{\Omega} [2uu_x \operatorname{sgn}_{\eta}(\theta - 2) + u^2 \operatorname{sgn}'_{\eta}(\theta - 2)] \frac{r^{2(n-1)} \theta_x}{v} \\
&\leq - \lim_{\eta \rightarrow 0+} \int_0^t \int_{\Omega} 2uu_x \operatorname{sgn}_{\eta}(\theta - 2) \frac{r^{2(n-1)} \theta_x}{v} \\
&\leq \frac{c_v}{8} \int_0^t \int_{\Omega_2(t)} \frac{r^{2(n-1)} \theta_x^2}{v} + C \int_0^t \int_{\Omega_2(t)} r^{2(n-1)} u^2 u_x^2 \\
&\leq \frac{c_v}{8} \int_0^t \int_{\Omega_2(t)} \frac{r^{2(n-1)} \theta_x^2}{v} + C \int_0^t \int_{\Omega_2(t)} u^2 (r^{n-1} u)_x^2 + C \int_0^t \max_{x \in \Omega_2(t)} u^4,
\end{aligned} \tag{4.12}$$

where the last inequality owes to (2.5), (1.18), and (3.1).

Notice from (1.16) that

$$(r^{n-2} u^2)_x = ur^{-1} [2(r^{n-1} u)_x - nr^{-1} vu].$$

This, along with (4.2), (1.18) and (3.1), brings to

$$\begin{aligned}
\int_0^t \int_{\Omega_2(t)} u^2 (r^{n-2} u^2)_x &= \int_0^t \int_{\Omega_2(t)} \left(2 \frac{u^3 (r^{n-1} u)_x}{r} - \frac{nu^4 v}{r^2} \right) \\
&\leq C \int_0^t \int_{\Omega} u^2 (r^{n-1} u)_x^2 + \int_0^t \max_{x \in \Omega_2(t)} u^4,
\end{aligned}$$

and therefore,

$$\begin{aligned}
\int_0^t \int_{\Omega_2(t)} u^2 \tilde{R} &\leq C \int_0^t \int_{\Omega} u^2 (r^{n-1} u)_x^2 + C \int_0^t \int_{\Omega} (\theta - 2)_+^2 u^2 \\
&\quad + C(\beta) \int_0^t \max_{x \in \Omega_2(t)} u^4(\cdot, t) + \beta c_v \int_0^t \int_{\Omega} \frac{(r^{n-1} u)_x^2}{v}.
\end{aligned} \tag{4.13}$$

Inequalities (4.12) and (4.13) guarantee that (4.11) satisfies

$$\begin{aligned}
I_5 &\leq C + \frac{\kappa}{8} \int_0^t \int_{\Omega} \frac{r^{2(n-1)} |\partial_x(\theta - 2)_+|^2}{v} + C \int_0^t \int_{\Omega} u^2 (r^{n-1} u)_x^2 \\
&\quad + C \int_0^t \left[\max_{x \in \Omega} (\theta - 2)_+^2 + \max_{x \in \Omega_2(t)} u^4 \right] + \beta \int_0^t \int_{\Omega} \frac{(r^{n-1} u)_x^2}{v}.
\end{aligned}$$

On account of the estimates for I_i ($i = 1 \sim 5$) above, it follows from (3.1) and (4.6) that

$$\begin{aligned}
& \int_{\Omega} (\theta - 2)_+^2 + \int_0^t \int_{\Omega} \left[(r^{n-1} u)_x^2 + (\theta - 2)_+ (r^{n-1} u)_x^2 + r^{2(n-1)} |\partial_x(\theta - 2)_+|^2 \right] \\
&\leq C + C \int_0^t \int_{\Omega} u^2 (r^{n-1} u)_x^2 + \varepsilon \int_0^t \int_{\Omega} \theta (r^{n-1} u)_x^2 \\
&\quad + C \int_0^t \left[\max_{x \in \Omega} (\theta - 2)_+^2 + \max_{x \in \Omega} u^4 + \max_{x \in \Omega_2(t)} (\theta - 1)^2 \right].
\end{aligned} \tag{4.14}$$

Noting from (2.1) and (2.5) that

$$\begin{aligned}
\int_0^t \int_{\Omega} r^{2(n-1)} \theta_x^2 &= \int_0^t \left(\int_{\Omega_2(t)} + \int_{\Omega \setminus \Omega_2(t)} \right) r^{2(n-1)} \theta_x^2 \\
&= C \int_0^t \int_{\Omega} r^{2(n-1)} |\partial_x(\theta - 2)_+|^2 + C \int_0^t \int_{\Omega \setminus \Omega_2(t)} r^{2(n-1)} \theta_x^2 \\
&\leq C \int_0^t \int_{\Omega} r^{2(n-1)} |\partial_x(\theta - 2)_+|^2 + C \int_0^t \int_{\Omega \setminus \Omega_2(t)} \frac{r^{2(n-1)} \theta_x^2}{v \theta^2} \\
&\leq C \int_0^t \int_{\Omega} r^{2(n-1)} |\partial_x(\theta - 2)_+|^2 + C
\end{aligned}$$

and that

$$\begin{aligned}
\int_0^t \int_{\Omega} \theta (r^{n-1} u)_x^2 &= \int_0^t \left(\int_{\Omega_3(t)} + \int_{\Omega \setminus \Omega_3(t)} \right) \theta (r^{n-1} u)_x^2 \\
&\leq 3 \int_0^t \int_{\Omega_3(t)} (\theta - 2)_+ (r^{n-1} u)_x^2 + 3 \int_0^t \int_{\Omega \setminus \Omega_3(t)} (r^{n-1} u)_x^2 \\
&\leq 3 \int_0^t \int_{\Omega_3(t)} (\theta - 2)_+ (r^{n-1} u)_x^2 + C,
\end{aligned}$$

we select ε in (4.14) so small such that

$$\begin{aligned}
&\int_{\Omega} (\theta - 2)_+^2 + \int_0^t \int_{\Omega} \left[(1 + \theta) (r^{n-1} u)_x^2 + r^{2(n-1)} \theta_x^2 \right] \\
&\leq C + C \int_0^t \int_{\Omega} u^2 (r^{n-1} u)_x^2 + C \int_0^t \left[\max_{x \in \Omega} (\theta - 2)_+^2 + \max_{x \in \Omega} u^4 + \max_{x \in \Omega_2(t)} (\theta - 1)^2 \right].
\end{aligned} \tag{4.15}$$

Step 2. Multiply (1.13)₂ by u^3 and use

$$(r^{n-1} u^3)_x = 3u^2 (r^{n-1} u)_x - \frac{2(n-1)}{r} v u^3, \tag{4.16}$$

we obtain

$$\begin{aligned}
&\frac{1}{4} \int_{\Omega} u^4(x, t) + 3\beta \int_0^t \int_{\Omega} \frac{u^2 (r^{n-1} u)_x^2}{v} \\
&= \frac{1}{4} \int_{\Omega} u_0^4 + 2\beta(n-1) \int_0^t \int_{\Omega} \frac{(r^{n-1} u)_x u^3}{r} - R \int_0^t \int_{\Omega} \left(\frac{\theta}{v} \right)_x r^{n-1} u^3.
\end{aligned} \tag{4.17}$$

By (2.1) and (1.18), one has

$$2\beta(n-1) \int_0^t \int_{\Omega} \frac{(r^{n-1} u)_x u^3}{r} \leq \varepsilon \int_0^t \int_{\Omega} (r^{n-1} u)_x^2 + C(\varepsilon) \int_0^t \max_{x \in \Omega} u^4(\cdot, t). \tag{4.18}$$

In view of (2.1), (3.1), and (4.2), we deduce

$$\begin{aligned}
&\int_0^t \int_{\Omega} \left[3u^2 (r^{n-1} u)_x \frac{\theta - 1}{v} - 2(n-1) \frac{\theta - 1}{r} u^3 \right] \\
&= \int_0^t \left(\int_{\Omega \setminus \Omega_2(t)} + \int_{\Omega_2(t)} \right) \left[3u^2 (r^{n-1} u)_x \frac{\theta - 1}{v} - 2(n-1) \frac{\theta - 1}{r} u^3 \right] \\
&\leq \varepsilon \int_0^t \int_{\Omega \setminus \Omega_2(t)} [(r^{n-1} u)_x^2 + 2(\theta - 1)^2 u^4] + C \int_0^t \int_{\Omega \setminus \Omega_2(t)} \left[(\theta - 1)^2 u^4 + \frac{v u^2}{r^2 \theta} \right] \\
&\quad + \varepsilon \int_0^t \int_{\Omega_2(t)} [u^2 (r^{n-1} u)_x^2 + (\theta - 1)^2] + C \int_0^t \int_{\Omega_2(t)} [(\theta - 1)^2 u^2 + u^6] \\
&\leq \varepsilon \int_0^t \int_{\Omega} (1 + u^2) (r^{n-1} u)_x^2 + C \int_0^t \left[\max_{\Omega_2(t)} (\theta - 1)^2 + \max_{x \in \Omega} u^4 \right] + C.
\end{aligned} \tag{4.19}$$

Similar argument, combing with (4.3), runs

$$\begin{aligned}
& \int_0^t \int_{\Omega} \left[3u^2(r^{n-1}u)_x \frac{1-v}{v} - 2(n-1)\frac{1-v}{r}u^3 \right] \\
& \leq \varepsilon \int_0^t \int_{\Omega} (r^{n-1}u)_x^2 + C \int_0^t \int_{\Omega} (1-v)^2 u^4 + C \int_0^t \left(\int_{\Omega \setminus \Omega_2(t)} + \int_{\Omega_2(t)} \right) \frac{|1-v|u^3}{r} \\
& \leq \varepsilon \int_0^t \int_{\Omega} (r^{n-1}u)_x^2 + C \int_0^t \int_{\Omega} (1-v)^2 u^4 \\
& \quad + C \int_0^t \int_{\Omega \setminus \Omega_2(t)} (v-1)^2 u^4 + C \int_0^t \int_{\Omega_2(t)} \theta(1-v)^2 u^4 + C \int_0^t \int_{\Omega} \frac{vu^2}{r^2\theta} \\
& \leq \varepsilon \int_0^t \int_{\Omega} (r^{n-1}u)_x^2 + C \int_0^t \max_{x \in \Omega} u^4 + C.
\end{aligned} \tag{4.20}$$

With the help of (4.19) and (4.20), we use (4.16) to estimate

$$\begin{aligned}
& -R \int_0^t \int_{\Omega} \left(\frac{\theta}{v} \right)_x (r^{n-1}u^3) \\
& = R \int_0^t \int_{\Omega} \left(\frac{\theta-1}{v} + \frac{1-v}{v} \right) (r^{n-1}u^3)_x \\
& = R \int_0^t \int_{\Omega} \left[3u^2(r^{n-1}u)_x \frac{\theta-1}{v} - 2(n-1)\frac{\theta-1}{r}u^3 \right] \\
& \quad + R \int_0^t \int_{\Omega} \left[3u^2(r^{n-1}u)_x \frac{1-v}{v} - 2(n-1)\frac{1-v}{r}u^3 \right] \\
& \leq \varepsilon \int_0^t \int_{\Omega} (1+u^2)(r^{n-1}u)_x^2 + C \int_0^t \left[\max_{\Omega_2(t)} (\theta-1)^2 + \max_{x \in \Omega} u^4 \right] + C.
\end{aligned} \tag{4.21}$$

Inequalities (4.18) and (4.21) ensure that (4.17) satisfies

$$\begin{aligned}
& \int_{\Omega} u^4 + \int_0^t \int_{\Omega} u^2 (r^{n-1}u)_x^2 \\
& \leq C + C\varepsilon \int_0^t \int_{\Omega} (r^{n-1}u)_x^2 + C \int_0^t \left(\max_{x \in \Omega} u^4 + \max_{x \in \Omega_2(t)} (\theta-1)^2 \right).
\end{aligned} \tag{4.22}$$

Multiplying (4.22) by a large constant, adding the resulting expression up to (4.15), choosing ε sufficiently small, we conclude

$$\begin{aligned}
& \int_{\Omega} [(\theta-2)_+^2 + u^4] + \int_0^t \int_{\Omega} [(1+\theta+u^2)(r^{n-1}u)_x^2 + r^{2(n-1)}\theta_x^2] \\
& \leq C + C \int_0^t \left[\max_{x \in \Omega} (\theta-2)_+^2 + \max_{x \in \Omega_2(t)} (\theta-1)^2 + \max_{x \in \Omega} u^4 \right] \\
& \leq C + C \int_0^t \max_{x \in \Omega} [(\theta-3/2)_+^2 + u^4].
\end{aligned} \tag{4.23}$$

Step 3. It remains to estimate the terms on the right hand side of (4.23). Utilizing (4.3) and (1.18) we compute

$$\begin{aligned}
(\theta(x,t) - 3/2)_+^2 &= -2 \int_x^\infty (\theta - 3/2)_+ \partial_y (\theta - 3/2)_+ \\
&\leq C \int_{\Omega_{3/2}(t)} (\theta - 3/2)_+ |\theta_x| \\
&\leq \frac{C}{\sqrt{\delta_1}} \int_{\Omega} \frac{\theta_x^2}{\theta} + \sqrt{\delta_1} \int_{\Omega_{3/2}(t)} (\theta - 3/2)_+^2 \theta \\
&\leq \sqrt{\delta_1} \int_{\Omega} r^{2(n-1)} \theta_x^2 + \frac{C}{\delta_1^{3/2}} \int_{\Omega} \frac{r^{2(n-1)} \theta_x^2}{v \theta^2} + C \sqrt{\delta_1} \max_{x \in \Omega} (\theta(\cdot, t) - 3/2)_+^2,
\end{aligned}$$

which satisfies if δ_1 is chosen small

$$\max_{x \in \Omega} (\theta(\cdot, t) - 3/2)_+^2 \leq \frac{\sqrt{\delta_1}}{1 - C\sqrt{\delta_1}} \int_{\Omega} r^{2(n-1)} \theta_x^2 + \frac{C}{\delta_1^{3/2}(1 - C\sqrt{\delta_1})} \int_{\Omega} \frac{r^{2(n-1)} \theta_x^2}{v \theta^2}.$$

This combine with (2.1) lead to

$$\int_0^t \max_{x \in \Omega} (\theta - 3/2)_+^2 \leq \sqrt{\delta_1} \int_0^t \int_{\Omega} r^{2(n-1)} \theta_x^2 + C(\delta_1). \quad (4.24)$$

Next, by (3.1), (1.18) and (4.3), one has

$$\begin{aligned} u^4(x, t) &= 4 \int_0^x u^3 u_y \leq \frac{C}{\delta_2} \int_{\Omega} \frac{r^{2(n-1)} u_x^2}{v \theta} + \delta_2 \int_{\Omega} \frac{u^6 \theta}{r^{2(n-1)}} \\ &\leq \frac{C}{\delta_2} \int_{\Omega} \frac{r^{2(n-1)} u_x^2}{v \theta} + \delta_2 \int_{\Omega_2(t)} u^6 \theta + \delta_2 \int_{\Omega \setminus \Omega_2(t)} u^6 \theta \\ &\leq \frac{C}{\delta_2} \int_{\Omega} \frac{r^{2(n-1)} u_x^2}{v \theta} + C \delta_2 \max_{x \in \Omega} u^6(\cdot, t) + C \delta_2 \max_{x \in \Omega} u^4(\cdot, t). \end{aligned}$$

Thus, for small δ_2 one has

$$\max_{x \in \Omega} u^4(\cdot, t) \leq \frac{C}{\delta_2(1 - C\delta_2)} \int_{\Omega} \frac{r^{2(n-1)} u_x^2}{v \theta} + \frac{C\delta_2}{(1 - C\delta_2)} \max_{x \in \Omega} u^6(\cdot, t). \quad (4.25)$$

On the other hand,

$$\begin{aligned} \max_{x \in \Omega} u^6(\cdot, t) &= 6 \int_0^x u^5 u_y = 6 \int_0^x \frac{u^5 [(r^{n-1} u)_x - (n-1)r^{-1}vu]}{r^{n-1}} \quad (\text{by (2.5)}) \\ &\leq 6 \int_0^x \frac{u^5 (r^{n-1} u)_x}{r^{n-1}} \\ &\leq \frac{C}{\sqrt{\delta_2}} \int_{\Omega} u^2 (r^{n-1} u)_x^2 + \sqrt{\delta_2} \int_{\Omega} u^8 \\ &\leq \frac{C}{\sqrt{\delta_2}} \int_{\Omega} u^2 (r^{n-1} u)_x^2 + C \sqrt{\delta_2} \max_{x \in \Omega} u^6(\cdot, t). \end{aligned}$$

Insert it back into (4.25) and utilize (2.1) give that

$$\int_0^t \max_{x \in \Omega} u^4(\cdot, t) \leq C(\delta_2) + C\sqrt{\delta_2} \int_0^t \int_{\Omega} u^2 (r^{n-1} u)_x^2. \quad (4.26)$$

In terms of (4.26) and (4.24), choosing δ_1 and δ_2 so small such that (4.23) satisfies

$$\int_{\Omega} [(\theta - 2)_+^2 + u^4] + \int_0^t \int_{\Omega} [(1 + \theta + u^2)(r^{n-1} u)_x^2 + r^{2(n-1)} \theta_x^2] \leq C,$$

where the C does not rely on t . By this and (2.1) implies

$$\int_{\Omega} (\theta - 1)^2 = \left(\int_{\Omega \setminus \Omega_3(t)} + \int_{\Omega_3(t)} \right) (\theta - 1)^2 \leq C + C \int_{\Omega_3(t)} (\theta - 2)_+^2 \leq C.$$

The last two inequalities give birth to (4.1), the required. \square

Lemma 4.2 *Let (v, u, θ) be the solution described in Theorem 1.1. There is some C independent of t such that*

$$\int_{\Omega} v_x^2(x, t) + \int_0^t \int_{\Omega} (1 + \theta) v_x^2 \leq C. \quad (4.27)$$

Proof. By (1.13)₁, rewriting (1.13)₂ as the form

$$\beta \left(\frac{v_x}{v} \right)_t = R \left(\frac{\theta}{v} \right)_x + r^{1-n} u_t,$$

which yields after multiplied by v_x/v ,

$$\frac{\beta}{2} \int_{\Omega} \frac{v_x^2}{v^2}(x, t) + R \int_0^t \int_{\Omega} \frac{\theta v_x^2}{v^3} = \frac{\beta}{2} \int_{\Omega} \frac{v_x^2}{v^2}(x, 0) + R \int_0^t \int_{\Omega} \frac{v_x \theta_x}{v^2} + \int_0^t \int_{\Omega} r^{1-n} u_t \frac{v_x}{v}. \quad (4.28)$$

Cauchy-Schwarz inequality, (2.1), (1.18), and (4.1) guarantee that

$$\begin{aligned} R \int_0^t \int_{\Omega} \frac{v_x \theta_x}{v^2} &\leq \frac{R}{2} \int_0^t \int_{\Omega} \frac{\theta v_x^2}{v^3} + C \int_0^t \int_{\Omega} \frac{\theta_x^2}{v \theta} \\ &\leq \frac{R}{2} \int_0^t \int_{\Omega} \frac{\theta v_x^2}{v^3} + C \int_0^t \int_{\Omega} \frac{r^{2(n-1)} \theta_x^2}{v \theta^2} + C \int_0^t \int_{\Omega} r^{2(n-1)} \theta_x^2 \\ &\leq \frac{R}{2} \int_0^t \int_{\Omega} \frac{\theta v_x^2}{v^3} + C. \end{aligned}$$

Thanks to (1.16), (1.19), and (2.1), it gives

$$\begin{aligned} &\int_0^t \int_{\Omega} r^{1-n} u_t \frac{v_x}{v} \\ &= \int_{\Omega} r^{1-n} u \frac{v_x}{v}(x, t) - \int_{\Omega} r^{1-n} u \frac{v_x}{v}(x, 0) - \int_0^t \int_{\Omega} r^{1-n} u \left(\frac{v_x}{v} \right)_t + (n-1) \int_0^t \int_{\Omega} r^{-n} u^2 \frac{v_x}{v} \\ &\leq C + \frac{\beta}{4} \int_{\Omega} \left| \frac{v_x}{v} \right|^2 - \int_0^t \int_{\Omega} r^{1-n} u \left(\frac{v_x}{v} \right)_t + (n-1) \int_0^t \int_{\Omega} r^{-n} u^2 \frac{v_x}{v}. \end{aligned}$$

Therefore, (4.28) satisfies

$$\frac{\beta}{4} \int_{\Omega} \left| \frac{v_x}{v} \right|^2 + \frac{R}{2} \int_0^t \int_{\Omega} \frac{\theta v_x^2}{v^3} \leq C - \int_0^t \int_{\Omega} r^{1-n} u \left(\frac{v_x}{v} \right)_t + (n-1) \int_0^t \int_{\Omega} r^{-n} u^2 \frac{v_x}{v}. \quad (4.29)$$

Making use of (1.13)₁, (1.18), (4.1), (3.1), and

$$(r^{1-n} u)_x = r^{2(1-n)} (r^{n-1} u)_x + 2(1-n) r^{1-2n} v u,$$

we estimate

$$\begin{aligned} - \int_0^t \int_{\Omega} r^{1-n} u \left(\frac{v_x}{v} \right)_t &= \int_0^t \int_{\Omega} (r^{1-n} u)_x \frac{(r^{n-1} u)_x}{v} \\ &= \int_0^t \int_{\Omega} r^{2(1-n)} \frac{(r^{n-1} u)_x^2}{v} + 2(1-n) \int_0^t \int_{\Omega} r^{1-2n} u (r^{n-1} u)_x \\ &\leq C \int_0^t \int_{\Omega} (1+\theta) (r^{n-1} u)_x^2 + C \int_0^t \int_{\Omega} \frac{v u^2}{r^2 \theta} \\ &\leq C. \end{aligned} \quad (4.30)$$

Remember that (3.1) and (3.10), one has

$$\int_0^t \int_{\Omega} \frac{v_x^2}{v^2} \leq C(\alpha_1, \alpha_2) \left(\int_0^t f(t) \int_{\Omega} \frac{v_x^2}{v^2} + \int_0^t \int_{\Omega} \frac{\theta v_x^2}{v^2} \right), \quad (4.31)$$

where $f(t)$ is taken from (3.11). By this, (3.16), and Cauchy-Schwarz inequality, we have

$$\begin{aligned} \int_0^t \int_{\Omega} r^{-n} u^2 \frac{v_x}{v} &\leq C \int_0^t \max u^4 \int_{\Omega} r^{-2n} + \frac{R}{4} \left(\int_0^t f(t) \int_{\Omega} \frac{v_x^2}{v^2} + \int_0^t \int_{\Omega} \frac{\theta v_x^2}{v^2} \right) \\ &\leq C + \frac{R}{4} \left(\int_0^t f(t) \int_{\Omega} \frac{v_x^2}{v^2} + \int_0^t \int_{\Omega} \frac{\theta v_x^2}{v^2} \right), \end{aligned} \quad (4.32)$$

where the last inequality comes from (3.16), (4.26) and (4.1). Substituting (4.32) and (4.30) into (4.29) arrives at

$$\int_{\Omega} \left| \frac{v_x}{v} \right|^2 + \int_0^t \int_{\Omega} \frac{\theta v_x^2}{v^3} \leq C + C \int_0^t f(t) \int_{\Omega} \frac{v_x^2}{v^2}.$$

Gronwall inequality concludes that

$$\int_{\Omega} v_x^2 + \int_0^t \int_{\Omega} \theta v_x^2 \leq C,$$

which together with (2.1) and (3.1) deduce from (4.31) that

$$\int_0^t \int_{\Omega} v_x^2 \leq C.$$

The proof is complete. \square

Lemma 4.3 *It holds that*

$$\int_{\Omega} u_x^2(x, t) + \int_0^t \int_{\Omega} r^{2(n-1)} u_{xx}^2 \leq C \left(1 + \max_{\Omega \times [0, t]} \theta \right), \quad (4.33)$$

where the C is independent of t .

Proof. Multiplying (1.13)₂ by $-u_{xx}$, a straight calculation shows

$$\begin{aligned} & \frac{1}{2} \partial_t u_x^2 + \beta \frac{r^{2(n-1)} u_{xx}^2}{v} \\ &= (u_x u_t)_x + \beta u_{xx} \left(r^{2(n-1)} \frac{v_x u_x}{v^2} + (n-1) \frac{uv}{r^2} - 2(n-1) r^{n-2} u_x \right) \\ &+ R u_{xx} r^{n-1} \left(\frac{\theta_x}{v} - \frac{\theta v_x}{v^2} \right). \end{aligned}$$

Applying Cauchy-Schwarz inequality, we integrate it to find

$$\begin{aligned} & \frac{1}{2} \int_{\Omega} u_x^2(x, t) + \beta \int_0^t \int_{\Omega} \frac{r^{2(n-1)} u_{xx}^2}{v} \\ & \leq \frac{1}{2} \int_{\Omega} u_{0x}^2 + \frac{\beta}{4} \int_0^t \int_{\Omega} \frac{r^{2(n-1)} u_{xx}^2}{v} \\ & + C \int_0^t \int_{\Omega} \left[r^{2(n-1)} v_x^2 u_x^2 + \frac{u^2}{r^{2(n+1)}} + \frac{u_x^2}{r^2} + \theta_x^2 + \theta^2 v_x^2 \right]. \end{aligned} \quad (4.34)$$

It follows from (3.1), (4.1), (4.27), (1.18) and (2.1) that

$$\begin{aligned} & C \int_0^t \int_{\Omega} \left(\frac{u^2}{r^{2(n+1)}} + \frac{u_x^2}{r^2} + \theta_x^2 + \theta^2 v_x^2 \right) \\ & \leq C \left(1 + \max_{\Omega \times [0, t]} \theta \right) \int_0^t \int_{\Omega} \left(\frac{v u^2}{r^2 \theta} + \frac{r^{2(n-1)} u_x^2}{v \theta} + r^{2(n-1)} \theta_x^2 + \theta v_x^2 \right) \\ & \leq C \left(1 + \max_{\Omega \times [0, t]} \theta \right). \end{aligned}$$

Since $H^1 \hookrightarrow L^\infty$, we use (4.27) and (2.1) to get

$$\begin{aligned} C \int_0^t \int_{\Omega} r^{2(n-1)} v_x^2 u_x^2 & \leq C \int_0^t \|r^{n-1} u_x\|_{L^\infty}^2 \int_{\Omega} v_x^2 \\ & \leq \frac{\beta}{4} \int_0^t \int_{\Omega} r^{2(n-1)} u_{xx}^2 + C \int_0^t \int_{\Omega} r^{2(n-1)} u_x^2 \\ & \leq \frac{\beta}{4} \int_0^t \int_{\Omega} r^{2(n-1)} u_{xx}^2 + C \max_{\Omega \times [0, t]} \theta. \end{aligned} \quad (4.35)$$

With the last two inequalities in hand, we conclude the desired (4.33) from (4.34). \square

Lemma 4.4 *It holds that*

$$\int_{\Omega} \theta_x^2(x, t) + \int_0^t \int_{\Omega} r^{2(n-1)} \theta_{xx}^2 \leq C \left(1 + \max_{\Omega \times [0, t]} \theta^2 \right), \quad (4.36)$$

where the C is independent of t .

Proof. Multiplying (1.13)₃ by $-\theta_{xx}$ brings to

$$\begin{aligned} & \frac{c_v}{2} \partial_t \theta_x^2 + \kappa \frac{r^{2(n-1)} \theta_{xx}^2}{v} \\ &= (c_v \theta_x \theta_t)_x + \kappa \theta_{xx} \left(r^{2(n-1)} \frac{v_x \theta_x}{v^2} - 2(n-1) r^{n-2} \theta_x \right) \\ &+ \theta_{xx} \left(R \frac{\theta}{v} (r^{n-1} u)_x + 2\mu(n-1) (r^{n-2} u^2)_x - \beta \frac{(r^{n-1} u)_x^2}{v} \right). \end{aligned}$$

By Cauchy-Schwarz inequality, (3.1), (2.5), as well as

$$(r^{n-2} u^2)_x = 2r^{-1} u (r^{n-1} u)_x - n r^{-2} u^2 v, \quad (4.37)$$

we see that

$$\begin{aligned} & \frac{c_v}{2} \int_{\Omega} \theta_x^2 + \kappa \int_0^t \int_{\Omega} \frac{r^{2(n-1)} \theta_{xx}^2}{v} \\ & \leq \frac{c_v}{2} \int_{\Omega} \theta_{0x}^2 + \frac{\kappa}{4} \int_0^t \int_{\Omega} \frac{r^{2(n-1)} \theta_{xx}^2}{v} + C \int_0^t \int_{\Omega} \frac{r^{2(n-1)} v_x^2 \theta_x^2}{v^3} \\ & + C \int_0^t \int_{\Omega} \left[\theta_x^2 + (\theta^2 + r^{-2} u^2) (r^{n-1} u)_x^2 + r^{-2(n+1)} u^4 + r^{2(n-1)} u_x^4 \right]. \end{aligned} \quad (4.38)$$

Owing to (4.27) and (4.1), a similar argument as (4.35) shows

$$\begin{aligned} C \int_0^t \int_{\Omega} r^{2(n-1)} v_x^2 \theta_x^2 & \leq \frac{\kappa}{4} \int_0^t \int_{\Omega} r^{2(n-1)} \theta_{xx}^2 + C \int_0^t \int_{\Omega} r^{2(n-1)} \theta_x^2 \\ & \leq \frac{\kappa}{4} \int_0^t \int_{\Omega} r^{2(n-1)} \theta_{xx}^2 + C. \end{aligned}$$

In terms of (1.18), (3.16), (4.26), and (4.1), it satisfies

$$\begin{aligned} & \int_0^t \int_{\Omega} \left(\theta_x^2 + (\theta^2 + r^{-2} u^2) (r^{n-1} u)_x^2 + r^{-2(n+1)} u^4 \right) \\ & \leq C \left(1 + \max_{\Omega \times [0, t]} \theta \right) \int_0^t \int_{\Omega} \left[r^{2(n-1)} \theta_x^2 + (\theta + u^2) (r^{n-1} u)_x^2 \right] + C \int_0^t \max_{x \in \Omega} u^4 \int_{\Omega} r^{-2(n+1)} \\ & \leq C \left(1 + \max_{\Omega \times [0, t]} \theta \right). \end{aligned}$$

Finally, from (4.33) and (4.35) we obtain

$$\begin{aligned} \int_0^t \int_{\Omega} r^{2(n-1)} u_x^4 & \leq C \int_0^t \|r^{n-1} u_x\|_{L^\infty}^2 \int_{\Omega} u_x^2 \\ & \leq C \max_{\Omega \times [0, t]} \theta \int_0^t \|r^{n-1} u_x\|_{L^\infty}^2 \leq C \left(1 + \max_{\Omega \times [0, t]} \theta^2 \right). \end{aligned}$$

Insert the last three inequalities guarantee that (4.38) receives the (4.36). \square

Corollary 4.5 (Bound of u and θ) *There exists some constant C such that*

$$|u(x, t)| + \theta(x, t) \leq C, \quad (x, t) \in \overline{\Omega} \times [0, \infty). \quad (4.39)$$

Proof. In view of (4.1) and (4.36), we use Sobolev inequality to get

$$\|\theta - 1\|_{L^\infty(\Omega)}^2 \leq C \|\theta - 1\|_{L^2(\Omega)} \|\theta_x\|_{L^2(\Omega)} \leq C \left(1 + \max_{\Omega \times [0, t]} \theta\right),$$

which means for some C independent of t , such that

$$\theta(x, t) \leq C, \quad (x, t) \in \overline{\Omega} \times [0, t]. \quad (4.40)$$

Once (4.40) is obtained, it follows from (2.1) and (4.33) that

$$|u(x, t)| \leq C, \quad (x, t) \in \overline{\Omega} \times [0, t].$$

The proof is done. \square

5 Estimates for derivatives

The lemmas in this section concern the derivatives estimates, which are needed to show the large-time behavior of solutions.

Lemma 5.1 *Let (v, u, θ) be the solution obtained in Theorem 1.1. Then there is some C independent of t , such that*

$$\int_{\Omega} r^{2(n-1)} v_x^2(x, t) + \int_0^t \int_{\Omega} (1 + \theta) r^{2(n-1)} v_x^2 \leq C. \quad (5.1)$$

Proof. Utilizing (1.16) and (1.13)₁, the (1.13)₂ takes the form

$$\beta \left(r^{n-1} \frac{v_x}{v} \right)_t + R r^{n-1} \frac{\theta v_x}{v^2} = R r^{n-1} \frac{\theta_x}{v} + u_t - \beta(n-1) r^{n-2} u \frac{v_x}{v}.$$

Multiplied by $r^{n-1} v_x / v$, it gives

$$\begin{aligned} & \frac{\beta}{2} \int_{\Omega} \left| r^{n-1} \frac{v_x}{v} \right|^2 + R \int_0^t \int_{\Omega} r^{2(n-1)} \frac{\theta v_x^2}{v^3} \\ & \leq \frac{R}{4} \int_0^t \int_{\Omega} r^{2(n-1)} \frac{\theta v_x^2}{v^3} + C \int_0^t \int_{\Omega} r^{2(n-1)} \frac{\theta_x^2}{v \theta} \\ & \quad + \int_0^t \int_{\Omega} u_t r^{n-1} \frac{v_x}{v} - \beta(n-1) \int_0^t \int_{\Omega} r^{2(n-1)-1} u \frac{v_x^2}{v^2}. \end{aligned} \quad (5.2)$$

It yields from (1.13)₁, (2.1), (4.1) and (4.39) that

$$\begin{aligned} & \int_0^t \int_{\Omega} u_t r^{n-1} \frac{v_x}{v} \\ & = \int_{\Omega} u r^{n-1} \frac{v_x}{v}(x, t) - \int_{\Omega} u r^{n-1} \frac{v_x}{v}(x, 0) \\ & \quad - (n-1) \int_0^t \int_{\Omega} u^2 r^{n-2} \frac{v_x}{v} + \int_0^t \int_{\Omega} \frac{(r^{n-1} u)_x^2}{v} \\ & \leq C + \frac{\beta}{4} \int_{\Omega} \left| r^{n-1} \frac{v_x}{v} \right|^2 + \frac{R}{8} \int_0^t \int_{\Omega} r^{2(n-1)} \frac{\theta v_x^2}{v^3}. \end{aligned} \quad (5.3)$$

By (1.18), (3.1) and (4.39), Sobolev inequality gives

$$\begin{aligned} \|r^{-1} u\|_{L^\infty}^2 & \leq C \left(\|u r^{-1}\|_{L^2(\Omega)}^2 + \|(u r^{-1})_x\|_{L^2(\Omega)}^2 \right) \\ & \leq C \int_{\Omega} \frac{v u^2}{r^2 \theta} + C \int_{\Omega} \frac{r^{2(n-1)} u_x^2}{v \theta} =: g(t). \end{aligned}$$

This combine with Cauchy-Schwarz inequality conclude that

$$\begin{aligned}
& -\beta(n-1) \int_0^t \int_{\Omega} r^{2(n-1)-1} u \frac{v_x^2}{v^2} \\
& \leq \varepsilon \int_0^t \int_{\Omega} r^{2(n-1)} \frac{v_x^2}{v^2} + C(\varepsilon) \int_0^t \|r^{-1}u\|_{L^\infty}^2 \int_{\Omega} r^{2(n-1)} \frac{v_x^2}{v^2} \\
& \leq \varepsilon C(\alpha_1, \alpha_2) \int_0^t \int_{\Omega} r^{2(n-1)} \frac{\theta v_x^2}{v^2} + C(\varepsilon) \int_0^t [f(t) + g(t)] \int_{\Omega} r^{2(n-1)} \frac{v_x^2}{v^2}.
\end{aligned} \tag{5.4}$$

where in the last inequality we have used (4.31).

If we choose ε sufficiently small such that $\varepsilon C(\alpha_1, \alpha_2) \leq R/8$, substitute (5.3) and (5.4) into (5.2), use (4.1), (3.1), (2.1) and Gronwall inequality, it provides that

$$\int_{\Omega} r^{2(n-1)} v_x^2 + \int_0^t \int_{\Omega} r^{2(n-1)} \theta v_x^2 \leq C.$$

This, again with (4.31), yields the desired (5.1). \square

Lemma 5.2 *Let (v, u, θ) be the solution obtained in Theorem 1.1. It holds that*

$$\int_{\Omega} (r^{n-1}u)_x^2(x, t) + \int_0^t \int_{\Omega} u_t^2 \leq C, \tag{5.5}$$

where the C is independent of t .

Proof. Multiplying (1.13)₂ by u_t gives rise to

$$\begin{aligned}
& \int_0^t \int_{\Omega} u_t^2 \\
& = -\beta \int_0^t \int_{\Omega} \frac{(r^{n-1}u)_x}{v} (r^{n-1}u_t)_x + R \int_0^t \int_{\Omega} \left(\frac{\theta v_x}{v^2} - \frac{\theta_x}{v} \right) r^{n-1} u_t \\
& \leq -\beta \int_0^t \int_{\Omega} \frac{(r^{n-1}u)_x}{v} (r^{n-1}u_t)_x + \frac{1}{2} \int_0^t \int_{\Omega} u_t^2 + C \int_0^t \int_{\Omega} r^{2(n-1)} (\theta_x^2 + \theta^2 v_x^2) \\
& \leq -\beta \int_0^t \int_{\Omega} \frac{(r^{n-1}u)_x}{v} (r^{n-1}u_t)_x + \frac{1}{2} \int_0^t \int_{\Omega} u_t^2 + C,
\end{aligned} \tag{5.6}$$

where in the last inequality we have used (4.1), (4.39) and (5.1).

In terms of (2.1), (1.18), (4.37), (3.1), (4.27) and (4.39), a straight computation shows

$$\begin{aligned}
& -\beta \int_0^t \int_{\Omega} \frac{(r^{n-1}u)_x}{v} (r^{n-1}u_t)_x \\
& = -\beta \int_0^t \int_{\Omega} \frac{(r^{n-1}u)_x}{v} [(r^{n-1}u)_{xt} - (n-1)(r^{n-2}u^2)_x] \\
& = \frac{\beta}{2} \int_{\Omega} \frac{(r^{n-1}u)_x^2}{v}(x, 0) - \frac{\beta}{2} \int_{\Omega} \frac{(r^{n-1}u)_x^2}{v}(x, t) \\
& \quad - \frac{\beta}{2} \int_0^t \int_{\Omega} \frac{(r^{n-1}u)_x^3}{v^2} + \beta(n-1) \int_0^t \int_{\Omega} \frac{(r^{n-1}u)_x}{v} (r^{n-2}u^2)_x \\
& \leq C - \frac{\beta}{2} \int_{\Omega} \frac{(r^{n-1}u)_x^2}{v}(x, t) + C \int_0^t \int_{\Omega} (r^{n-1}u)_x^4.
\end{aligned}$$

By this and (3.1), we conclude from (5.6) that

$$\int_0^t \int_{\Omega} u_t^2 + \int_{\Omega} (r^{n-1}u)_x^2 \leq C + C \int_0^t \|(r^{n-1}u)_x\|_{L^\infty}^2 \int_{\Omega} (r^{n-1}u)_x^2. \tag{5.7}$$

Observe from (1.18), (2.5), (4.27), (4.33), (2.1) and (4.39) that

$$\begin{aligned}
\int_0^t \|(r^{n-1}u)_x\|_{L^\infty}^2 &\leq C \int_0^t \left(\|(r^{n-1}u)_{xx}\|_{L^2(\Omega)}^2 + \|(r^{n-1}u)_x\|_{L^2(\Omega)}^2 \right) \\
&\leq C \int_0^t \int_\Omega \left(r^{2(n-1)}u_{xx}^2 + v_x^2 + \frac{r^{2(n-1)}u_x^2}{v\theta} + \frac{vu^2}{r^2\theta} \right) \\
&\leq C,
\end{aligned} \tag{5.8}$$

we apply Gronwall inequality to (5.7) to receive the (5.5). \square

Lemma 5.3 *Let (v, u, θ) be the solution obtained in Theorem 1.1. Then it holds that*

$$\int_\Omega r^{2(n-1)}\theta_x^2(x, t) + \int_0^t \int_\Omega \theta_t^2 \leq C, \tag{5.9}$$

where the C is independent of t .

Proof. Multiplied by θ_t , it gives from (1.13)₃ that

$$\begin{aligned}
&c_v \int_0^t \int_\Omega \theta_t^2 + \frac{\kappa}{2} \int_\Omega \frac{r^{2(n-1)}\theta_x^2}{v} \\
&= \frac{\kappa}{2} \int_\Omega \frac{r^{2(n-1)}\theta_x^2}{v}(x, 0) + \frac{\kappa}{2} \int_0^t \int_\Omega \left(2(n-1) \frac{r^{2(n-1)}r^{-1}u\theta_x^2}{v} - \frac{r^{2(n-1)}\theta_x^2 v_t}{v^2} \right) \\
&\quad + \beta \int_0^t \int_\Omega \frac{(r^{n-1}u)_x^2 \theta_t}{v} - R \int_0^t \int_\Omega \frac{\theta}{v} (r^{n-1}u)_x \theta_t - 2(n-1)\mu \int_0^t \int_\Omega (r^{n-2}u^2)_x \theta_t,
\end{aligned}$$

which implies from Cauchy-Schwarz inequality, (3.1), (4.1), (4.39), (1.18), (1.13)₁, (2.1), (4.37), and (5.5) that

$$\int_0^t \int_\Omega \theta_t^2 + \int_\Omega r^{2(n-1)}\theta_x^2 \leq C + C \int_0^t \|(r^{n-1}u)_x\|_{L^\infty(\Omega)}^2 \left(1 + \int_\Omega r^{2(n-1)}\theta_x^2 \right).$$

Using Gronwall inequality and (5.8) to the above inequality completes the proof. \square

6 Proof of Theorem 1.2

This final section is devoted to proving Theorem 1.2.

Firstly, the (1.22) follows from inequality (2.1), Lemma 3.1, Lemma 4.1, Lemmas 5.1-5.3, and equations (1.13).

Next to prove (1.24), for this we verify

$$\int_0^\infty \left| \frac{d}{dt} \|v_x\|_{L^2(\Omega)}^2 \right| + \left| \frac{d}{dt} \|u_x\|_{L^2(\Omega)}^2 \right| + \left| \frac{d}{dt} \|\theta_x\|_{L^2(\Omega)}^2 \right| dt \leq C. \tag{6.1}$$

In fact, by (1.18), Lemmas 4.3-4.4, Lemmas 5.2-5.3, we compute

$$\begin{aligned}
&\int_0^\infty \left| \frac{d}{dt} \|u_x\|_{L^2(\Omega)}^2 \right| + \left| \frac{d}{dt} \|\theta_x\|_{L^2(\Omega)}^2 \right| dt \\
&= 2 \int_0^\infty \left| \int_\Omega u_x u_{xt} \right| + \left| \int_\Omega \theta_x \theta_{xt} \right| dt \\
&\leq \int_0^\infty \left(\|u_{xx}\|_{L^2(\Omega)}^2 + \|u_t\|_{L^2(\Omega)}^2 + \|\theta_{xx}\|_{L^2(\Omega)}^2 + \|\theta_t\|_{L^2(\Omega)}^2 \right) dt \\
&\leq C.
\end{aligned} \tag{6.2}$$

By (1.13)₁, (4.27) and (5.8), one has

$$\begin{aligned} \int_0^\infty \left| \frac{d}{dt} \|v_x\|_{L^2(\Omega)}^2 \right| dt &= 2 \int_0^\infty \left| \int_\Omega v_x (r^{n-1}u)_{xx} \right| dt \\ &\leq C \int_0^\infty \left(\|v_x\|_{L^2(\Omega)}^2 + \|(r^{n-1}u)_{xx}\|_{L^2(\Omega)}^2 \right) dt \\ &\leq C. \end{aligned} \quad (6.3)$$

Combination of (6.2) with (6.3) generates (6.1). On the other hand, it satisfies from (4.27), (4.1), (2.1), (3.1), (4.39) and (1.18) that

$$\int_0^\infty \left(\|v_x\|_{L^2(\Omega)}^2 + \|u_x\|_{L^2(\Omega)}^2 + \|\theta_x\|_{L^2(\Omega)}^2 \right) dt \leq C,$$

which together with (6.1) conclude that

$$\lim_{t \rightarrow \infty} \|(v_x, u_x, \theta_x)(\cdot, t)\|_{L^2(\Omega)} = 0. \quad (6.4)$$

Therefore, the desired (1.24) is a direct consequence of (6.4), (2.1), (3.1) and (4.1).

It is only left to check (1.23). Thanks to (3.1) and (4.39), it suffice to show $\theta(x, t)$ has a positive bound from below. For one hand, (1.24) implies there is some large time point T_1 such that

$$\theta(x, t) \geq 1/2, \quad \forall (x, t) \in \overline{\Omega} \times [T_1, \infty). \quad (6.5)$$

For another hand, it satisfies from [6, eq.(4.9)] that

$$\theta(x, t) \geq C(T_1), \quad (x, t) \in \overline{\Omega} \times [0, T_1]. \quad (6.6)$$

Combination of (6.5) with (6.6) proves the (1.23), and thus, the Theorem 1.2 is completed. \square

References

- [1] R. Adams: *Sobolev spaces*, New York: Academic Press, 1975.
- [2] S. Antontsev; A. Kazhikhov; V. Monakhov: *Boundary Value Problems in Mechanics of Nonhomogeneous Fluids*, Amsterdam, New York: North-Holland, 1990.
- [3] G. Batchelor: *An introduction to Fluid Dynamics*, London: Cambridge Univ. Press, 1967.
- [4] D. Hoff: *Global well-posedness of the Cauchy problem for the Navier-Stokes equations of nonisentropic flow with discontinuous initial data*, J. Diff. Eqns., 95 (1992), 33-74.
- [5] S. Jiang: *Large-time behavior of solutions to the equations of a viscous polytropic ideal gas*, Annli Mat. Pura Appl., 175 (1998), 253-275.
- [6] S. Jiang: *Global spherically symmetric solutions to the equations of a viscous polytropic ideal gas in an exterior domain*, Comm. Math. Phys., **178** (1996), 339-374.
- [7] S. Jiang: *Large-time behavior of solutions to the equations of a one-dimensional viscous polytropic ideal gas in unbounded domains*, Comm. Math. Phys., **200** (1999), 181-193.
- [8] S. Jiang: *Remarks on the asymptotic behaviour of solutions to the compressible Navier-Stokes equations in the half-line*, Proc. Roy. Soc. Edinb. Sect. A, **132**(2002), 627-638.
- [9] Y. Kanel: *Cauchy problem for the equations of gasdynamics with viscosity*, Siberian Math. J., 20 (1979), 208-218.
- [10] J. Li; Z. Liang: *Some uniform estimates and large-time behavior for one-dimensional compressible Navier-Stokes system in unbounded domains with large data*,

- [11] T. Liu; Y. Zeng: *Large time behavior of solutions for general quasilinear hyperbolic-parabolic systems of conservation laws*, Memoirs of the American Mathematical Society, no. 599 (1997).
- [12] A. Matsumura: *Large-time behavior of the spherically symmetric solutions of an isothermal model of compressible viscous gas*, Transport Theory Statist. Phys. 21 (1992) (Proceedings of the Fourth International Workshop on Mathematical Aspects of Fluid and Plasma Dynamics (Kyoto, 1991)), 579C592.
- [13] A. Matsumura; T. Nishida: *Initial-boundary value problems for the equations of motion of compressible viscous and heat-conductive fluids*, Comm. Math. Phys. 89 (1983), 445-464.
- [14] T. Nakamura; S. Nishibata: *Large-time behavior of spherically symmetric flow of heat-conductive gas in a field of potential forces*, India. Univ. Math. J., 2008, **57**(2) 1019-1054.
- [15] M. Okada; S. Kawashima: *On the equations of one-dimensional motion of compressible viscous fluids*, J. Math. Kyoto Univ., 23 (1983), 55-71.
- [16] Y. Qin: *Nonlinear Parabolic-Hyperbolic Coupled Systems and Their Attractors, Operator Theory, Advances and Applications*, Vol 184. Basel, Boston, Berlin: Birkhauser, 2008.
- [17] N. Itaya: *On a certain temporally global solution, spherically symmetric, for the compressible N - S equations*, The Jinbun ronshu of Kobe Univ. Commun., 21 (1985), 1-10. (Japanese)